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Published online: 31 Aug 2018.

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A New Method for Parameterization of General Type-2 Fuzzy Sets

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ABSTRACT

In this paper a new method for the parameterization of general type-2 fuzzy membership functions. The proposed method describes the methodology, equations and pseudo-code for building a set of general type-2 membership functions, which are a combination of two Gaussian-type primary membership functions (Gaussian with uncertain mean, and Gaussian with uncertain standard deviation), with multiple combinations of secondary membership functions (Gaussian, double Gaussian, general bell and trapezoidal). In addition, several application cases are used to illustrate the advantages of the proposed parameterization of general type-2 fuzzy sets; where the membership functions are designed using the parameterization approach and the general type-2 inference system is approximated using the α-planes theory. Simulation results illustrate that the parameterization offers an efficient way to represent these fuzzy sets. The main idea of the approach is to facilitate the use of general type-2 fuzzy systems in real world applications. The main contribution is a proposed new form of parameterizing general type-2 fuzzy sets that simplifies the efficient design of this type of sets.

1. Introduction

Fuzzy logic provides the ability of modeling uncertainty, vagueness and imprecision present in the vast majority of real world problems. It has found successful applications in a wide variety of fields, such as decision making [1–4], control system design [5–8], data classification [9–12], decision analysis [13–15], expert systems [16–18], time-series prediction [19–21], robotics [22–24], pattern recognition [10,25,26] and so on.

With ongoing research being done on fuzzy sets (FSs), improvements have appeared which best represent the true meaning of the original idea of linguistic variables, i.e. imprecision and uncertainty in linguistic variables. This idea has sprung three main representations of FSs, type-1 fuzzy sets (T1 FSs), interval type-2 fuzzy sets (IT2 FSs) and general type-2 fuzzy sets (GT2 FSs). In this context, T1 FSs are the simplest form of linguistic variable

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representation, and as such can only characterize a certain degree of imprecision, or ambiguity. With IT2 FSs the notion of uncertainty was introduced in the form of intervals. These intervals would ideally represent an infinite amount of embedded T1 FSs. Although more computationally complex when compared to a T1 FSs, when inferred upon, they improve the general fuzzy model by being more resilient to external noise. IT2 FSs have been used in multiple areas of applications, such as spatial analysis [27], mobile field workforce area optimization [28], analysis of failure modes [29], wing rock sliding controller [30]. The other main forms of representing FSs are GT2 FSs, which similarly to the IT2 FSs also intrinsically manage uncertainty. Where instead of representing uncertainty through an area, it is represented by a volume. This form of uncertainty representation, in essence, is much more resilient to noise than the IT2 FSs. Although research in this form of FS is fairly recent, some applications already exist, e.g. multi-central clustering [31], analysis of gene expression data [32], control of a mobile robot [33], image processing systems [34].

Fuzzy logic is widely applied in many areas because not only it can deal with incomplete or uncertain data, but also because its tools have been simplified by using parameterized FSs. This parameterization has been mainly used on the most common forms of membership functions, since the initial days of T1 FSs, e.g. Gaussian, Triangular, Rectangular; with IT2 FSs the same forms of parameterization have been directly implemented, albeit with intervals, since they can easily be transitioned to a higher type of FS is much easier for researchers to implement, by staying with what is familiar. This fact is what has kept a consistency throughout T1 and IT2FSs and has allowed researchers to perform direct comparisons between the performances of either Type of FS implementations. Even though there are parameterized membership functions, which have existed and have been used for a long time, new ones have been proposed, although not too widely used, but they too can be easily be transitioned from a type-1 parameterization to a type-2 parameterization. In short, parameterization has kept the implementation and ongoing research on fuzzy logic advancing, by letting researchers focus on the problem to model, and not on how to represent their FSs.

The main contribution of the paper is the proposed approach for the parameterization of general type-2 fuzzy sets, where the membership function is formed from given parameters which represent the support of the primary membership function, such that all secondary membership functions are automatically calculated in a continuous space. The importance of a good parameterization is to provide better tools to help efficiently design type-2 fuzzy systems. It is important to mention that this work is focused only in the general type-2 membership function parameterization. In this paper, the general type-2 inference system is approximated using α-planes.

There exists a previous implementation of GT2 FS parameterization [35], where the parameterization is for three types of primary membership functions (triangular, Gaussian and trapezoidal). In comparison, the proposed parameterization in this paper describes the process for building two Gaussian-type primary membership functions (Gaussian with uncertain mean and Gaussian with uncertain standard deviation), with multiple combinations of secondary membership functions (Gaussian, double Gaussian, general bell and trapezoidal). This with the addition of two extra parameters, which give better control over the amount of uncertainty in the support and the core of the parameterized GT2 membership function. Although enough description is also given so that other types of primary membership functions and/or secondary membership functions could be easily adapted.

The rest of the paper is organized as follows. Some basic concepts about GT2 FSs are defined in Section 2; afterwards, the methodology, equations and pseudo-code to obtain the general type-2 parameterization approach is presented in Section 3 and simulations results for three different applications: Mackey–Glass chaotic series prediction, water tank controller and the wave equation are presented in Section 4; the membership functions for these applications are designed using the parameterization approach and the general type-2 inference system is approximated using the α-planes theory. Finally, Section 5 concludes the paper with some remarks about the contribution.

2. Background

A T1 FS has been the first and most popular fuzzy concept [36,37]; advances in research of theory and practice around this area have made it possible to apply more complex forms of fuzzy logic, such as IT2 and GT2 FS. In addition, GT2 FSs have outperformed the IT2 and the T1 FSs in many real world applications. This is because a GT2 FS offers a way to model higher levels of uncertainty because of additional degrees of freedom provided by its third dimension; however, GT2 FSs are computationally more complex than T1 and IT2 FSs.

In the following sub-sections, we define some important concepts about GT2 FSs theory, which are used in the remainder of this paper.

2.1. General Type-2 Fuzzy Sets

A general type-2 fuzzy set denoted as $\tilde{\tilde{A}}$, on a universe of discourse X, can be expressed by (1); where $\mu_{\tilde{\tilde{A}}}(x, u)$ is a 3D membership function (Figure 1), $x \in X$ and $u \in J_x^u$ [5,38],

$$\tilde{\tilde{A}} = \left\{ \left((x, u), \mu_{\tilde{\tilde{A}}}(x, u) \right) \; \forall x \in X, \; \forall u \in J_x^u \subseteq [0, 1] \right\}. \tag{1}$$

Figure 1. Various elements of a general type-2 membership function.

In (1), x is the primary variable, u denotes the secondary variable, J_x^u represents an interval between the lower and the upper membership functions, and the secondary membership function is given by $\mu_{\tilde{\tilde{A}}}(x, u)$; where, $0 \leq \mu_{\tilde{\tilde{A}}}(x, u) \leq 1$. An alternative form of representation of a GT2 fuzzy set is given in (2), where $\int\int$ represents the union over the entire possible values of x, u and $\mu_{\tilde{A}}(x, u)$, [38]

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_{x \subseteq [0,1]}^u} \mu_{\tilde{A}}(x, u)/(x, u), \quad J_{x \subseteq [0,1]}^u. \tag{2}$$

In general, there are two important representations for GT2 FSs in which are included the vertical slice and the wavy slice representation. At each point of x, say $x = x'$, the 2-D plane, whose axes are u and $\mu_{\tilde{\tilde{A}}}(x', u)$ is called the vertical slice of $\mu_{\tilde{\tilde{A}}}(x, u)$. Symbolically, it is $\mu_{\tilde{\tilde{A}}}(x = x', u)$, for $x' \in X$ and $\forall_u \in J_x' \subseteq [0, 1]$, and it is described in (3),

$$\mu_{\tilde{\tilde{A}}}(x = x', u) = \int_{u \in J_{x'}} f_{x'}(u)/uJ_{x'} \subseteq [0, 1], \tag{3}$$

where $f_{x'}(u)$ is the amplitude of the secondary membership function and $f_{x'}(u) \subseteq [0, 1]$.

Uncertainty in the primary membership of a GT2 fuzzy set $\tilde{\tilde{A}}$ is represented by a bounded region; therefore, the two-dimensional support of $\mu_{\tilde{\tilde{A}}}(x, u)$ is called the footprint of uncertainty (FOU) of $\tilde{\tilde{A}}$ and is denoted by (4),

$$FOU\left(\tilde{\tilde{A}}\right) = \left\{(x, u) \in X \times [0, 1] \, \mu_{\tilde{\tilde{A}}}(x, u) > 0\right\}. \tag{4}$$

$FOU(\tilde{\tilde{A}})$ can also be expressed as the union of all primary memberships, i.e.

$$FOU\left(\tilde{\tilde{A}}\right) = \bigcup_{x \in X} J_x^u. \tag{5}$$

2.2. General Type-2 Fuzzy Systems

The general type-2 Mamdani fuzzy systems contain five components: fuzzifier, rules, inference engine, type-reducer and defuzzifier; these are interconnected as shown in Figure 2.

The fuzzifier process maps crisp numbers into GT2 FS. Then, we need to activate rules that are in terms of linguistic variables, which have fuzzy sets associated with them. The rules can be provided by experts or can be extracted from numerical data. In either case, the rules that we are interested in can be expressed as a collection of IF-THEN statements. Consider a GT2 FSs having the fuzzy sets $\tilde{F}$ and $\tilde{G}$ with p inputs $x_1 \in X_1, \ldots, x_p \in X_p$, one output $y \in Y$ and M rules, where the lth rule is expressed in (6),

$$R^l : \text{IF } x_1 \text{ is } \tilde{F}_1^l \text{ and } \cdots \text{ and } x_p \text{ is } \tilde{F}_p^l, \text{ THEN } y \text{ is } \tilde{G}^l, \text{ where } l = 1, \ldots, M. \tag{6}$$

The fuzzy inference engine process of a GT2 FSs can be simplified into two main operations, meet and join, as shown in (7) and (8),

$$\mu_{\tilde{\tilde{A}}}(x) \sqcup \mu_{\tilde{\tilde{B}}}(x) = \left\{\left[\int_{u \in J_x^u} \int_{u \in J_x^w} f_x(u) * g_x(w)/(u \vee w)\right]\right\}, \tag{7}$$
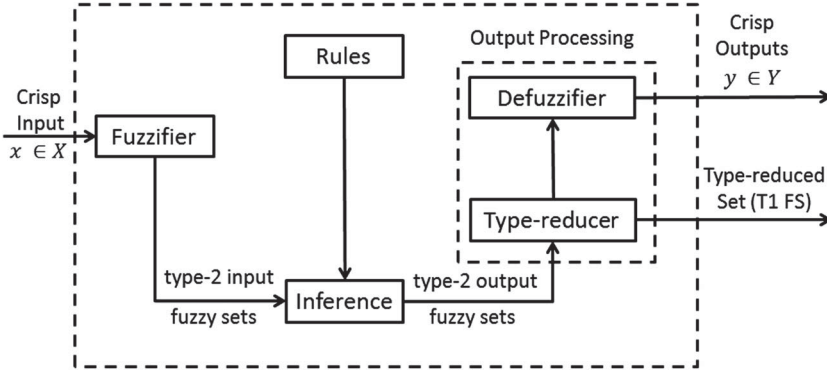
Figure 2. General structure of type-2 fuzzy logic system.

$$\mu_{\tilde{A}}(x) \sqcap \mu_{\tilde{B}}(x) = \left\{ \left[\int_{u \in J_x^u} \int_{u \in J_x^w} f_x(u) * g_x(w) / (u \wedge w) \right] \right\}. \tag{8}$$

The centroid is one of the techniques used in the type-reducer process for the GT2 FSs. The centroid definition $C_{\tilde{A}}$ of a GT2 FSs, introduced by Karnik and Mendel [39–41], is expressed in (9); where θ_i is a combination associated to the secondary degree $f_{x_1}(\theta_1) \tilde{*} \cdots \tilde{*} f_{x_N}(\theta_N)$,

$$C_{\tilde{A}} = \int_{\theta_1 \in J_{x_1}} \cdots \int_{\theta_N \in J_{x_N}} \left[f_{x_1}(\theta_1) \tilde{*} \cdots \tilde{*} f_{x_N}(\theta_N) \right] \Big/ \frac{\sum_{i=1}^{N} x_i \theta_i}{\sum_{i=1}^{N} \theta_i}. \tag{9}$$

2.3. α-Plane Representation of General Type-2 Fuzzy Sets

The GT2 FSs are computationally more complex than T1 and IT2 FSs; especially the defuzzifier process, which is an extremely costly operation and is not practical for most real world applications. Recently a range of alternative methods have been put forward to approximate the defuzzifier process; some of them are the α-planes and zSlices approaches [42]. These approximation techniques decompose the three-dimensional GT2 membership function by using different kinds of cuts to obtain a collection of IT2 FSs.

The term α-planes was introduced by Liu in 2008 [39]. An α-plane for a GT2 FS, is denoted by $\tilde{A}_{\alpha}$. It is the union of all primary memberships functions of $\tilde{A}$ whose secondary grades are greater than or equal to $\alpha(0 \leq \alpha \leq 1)$. The α-planes are represented in (10),

$$\tilde{A}_{\alpha} = \left\{ (x, u), \mu_{\tilde{A}}(x, u) \geq \alpha \mid \forall x \in X, \, \forall u \in [0, 1] \right\}$$

$$= \int_{\forall x \in X} \int_{\forall u \in [0,1]} \{(x, u) \mid f_x(u) \geq \alpha\}. \tag{10}$$

The union of all α-planes is expressed in (11); where $R_{\tilde{A}_{\alpha}}$ is one horizontal slice at level α.

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} R_{\tilde{A}_{\alpha}}. \tag{11}$$

3. On the Parameterization of a GT2 FS

For the proposed GT2 FS parameterization, primary membership function supports and secondary membership functions must be first defined. Once defined, any possible combination is possible.

3.1. Primary Membership Function Support Definitions

Two examples of primary membership function supports are presented, which are Gaussian membership functions with uncertain mean, and with uncertain standard deviation.

3.1.1. Gaussian Membership Function with Uncertain Mean

This support type is formed via (12) and (13), where $x \in X$ and X is the Universe of Discourse, m_1 and m_2 are the means of two T1 membership functions used to construct the GT2 FS support, σ is the standard deviation used by both T1 membership functions. $\mu_1(x)$ and $\mu_2(x)$ are the left and right membership functions, respectively, which will form the GT2 FS support,

$$\mu_1(x) = \exp\left[-\frac{1}{2}\left(\frac{x-m_1}{\sigma}\right)^2\right], \tag{12}$$

$$\mu_2(x) = \exp\left[-\frac{1}{2}\left(\frac{x-m_2}{\sigma}\right)^2\right]. \tag{13}$$

Once obtaining the left mean and right mean ($\mu_1(x)$ and $\mu_2(x)$) Gaussian membership functions, (14) and (15) are used to construct the support of the GT2 FS, where $\bar{\mu}(x)$ and $\underline{\mu}(x)$ are upper and lower membership functions, respectively,

$$\bar{\mu}(x) = \begin{cases} \mu_1(x), & x < m_1, \\ 1, & m_1 \le x \le m_2, \\ \mu_2(x), & x > m_2, \end{cases} \tag{14}$$

$$\underline{\mu}(x) = \begin{cases} \mu_2(x), & x \le \dfrac{m_1 + m_2}{2}, \\ \mu_1(x), & x > \dfrac{m_1 + m_2}{2}. \end{cases} \tag{15}$$

3.1.2. Gaussian Membership Function with Uncertain Standard Deviation

This support type is formed by Equations (16) and (17), where $\bar{\mu}(x)$ and $\underline{\mu}(x)$ are upper and lower membership functions, respectively, $x \in X$ and X is the Universe of Discourse, σ_1 and σ_2 are the standard deviations of two T1 membership functions used to construct the GT2 FS support, $\sigma_1 < \sigma_2$, and m is the mean used by both T1 membership functions.

$$\underline{\mu}(x) = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma_1}\right)^2\right], \tag{16}$$

$$\bar{\mu}(x) = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma_2}\right)^2\right]. \tag{17}$$

3.2. Baseline Secondary Membership Function Definitions

As a baseline to construct any secondary membership function, a Gaussian membership function is first required as a reference. This baseline secondary membership function is linked to the chosen primary membership function type, either Gaussian with uncertain mean, or Gaussian with uncertain standard deviation, and later used to obtain the parameters required by the secondary membership functions.

3.2.1. Based on a Gaussian Primary Membership Function with Uncertain Mean

The process of obtaining a baseline secondary membership function when the primary membership function is Gaussian with uncertain mean is performed via (18)–(20), where $x \in X$, p_x is a secondary Gaussian membership function, m_1 and m_2 are the left and right means of the primary membership function, σ is the standard deviation of the primary membership function, δ is the spread of the core (the core is represented by all values where $\mu(x) = 1$), σ_u [43] is the standard deviation of the baseline secondary Gaussian membership function, and ε is a very small threshold, used to prevent a near zero standard deviation σ.

$$p_x = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]; \quad \text{where} \quad m = \frac{m_1 + m_2}{2}, \tag{18}$$

$$\delta = \bar{\mu}(x) - \underline{\mu}(x), \tag{19}$$

$$\sigma_\mu = \frac{\delta}{2\sqrt{3}} + \varepsilon. \tag{20}$$

3.2.2. Based on a Gaussian Primary Membership Function with Uncertain Standard Deviation

The process of obtaining a baseline secondary membership function when the primary membership function is Gaussian with uncertain standard deviation is described by (21)–(23), where $x \in X$, p_x is a secondary Gaussian membership function, σ_1 and σ_2 are the inner and outer standard deviations of the primary membership function, σ is the standard deviation of the primary membership function, δ is the spread of the core (the core is represented by all values where $\mu(x) = 1$), σ_u [43] is the standard deviation of the secondary Gaussian membership function, ε is a very small threshold, used to prevent a near zero standard deviation σ.

$$p_x = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right]; \quad \text{where} \quad \sigma = \frac{\sigma_1 + \sigma_2}{2}, \tag{21}$$

$$\delta = \bar{\mu}(x) - \underline{\mu}(x), \tag{22}$$

$$\sigma_\mu = \frac{\delta}{2\sqrt{3}} + \varepsilon. \tag{23}$$

3.3. Secondary Membership Function Definitions

The secondary membership function can be chosen in accordance to the desired uncertainty form of representation. This paper introduces the use of four different secondary membership functions: Gaussian, double Gaussian, generally bell-shaped and trapezoidal.

3.3.1. Gaussian Membership Function

The Gaussian secondary membership function, shown in Figure 3 is obtained via (25), where $\tilde{u}(x, u)$ is a membership function which takes on the parameters $x \in X$ on the Universe of Discourse, and $u \in U$ such that $u \in J_x \subseteq [0, 1]$. Here p_x is obtained from the baseline Gaussian secondary membership function by (18) or (21), depending on the chosen primary membership function; and σ_u is obtained via (24).

$$\sigma_\mu = (1 - \rho) \frac{\delta}{2\sqrt{3}} + \varepsilon, \tag{24}$$

$$\tilde{\mu}(x, u) = \exp\left[-\frac{1}{2} \left(\frac{u - p_x}{\sigma_u} \right)^2 \right]. \tag{25}$$

3.3.2. Double Gaussian Membership Function

Double Gaussian secondary membership function parameters are defined by (26)–(29), where m_1 and σ_1 are the mean and standard deviation for the left Gaussian membership function, m_2 and σ_2 are the mean and standard deviation for the right Gaussian membership function. Both sets of parameters are affected by the fractions of uncertainty of λ and ρ, where the core and support are affected respectively, which ultimately affect the amount of uncertainty which is desired, where values of zero represent no change in uncertainty, and the value of 1 represents 100% extra uncertainty. These parameters, p_x and σ_u, are obtained by (18) and (20), or (21) and (23), depending on the chosen primary membership function.

$$m_1 = p_x (1 - \lambda), \tag{26}$$

Figure 3. Sample secondary Gaussian membership function.

Figure 4. Sample secondary double Gaussian membership function.

$$\sigma_1 = \sigma_u (1 - \rho), \tag{27}$$

$$m_2 = p_x (1 + \lambda), \tag{28}$$

$$\sigma_2 = \sigma_u (1 + \rho). \tag{29}$$

The double Gaussian secondary membership function, shown in Figure 4 is calculated via (30), where $\tilde{u}(x, u)$ is a membership function which takes on the parameters $x \in X$ on the Universe of Discourse, and $u \in U$ such that $u \in J_x \subseteq [0, 1]$,

$$\tilde{\mu}(x, u) = \begin{cases} \exp\left[-\frac{1}{2}\left(\frac{u - m_1}{\sigma_1}\right)^2\right], & u \leq c_1, \\ 1, & c_1 \leq u \leq c_2, \\ \exp\left[-\frac{1}{2}\left(\frac{u - m_2}{\sigma_2}\right)^2\right], & u \geq c_1. \end{cases} \tag{30}$$

3.3.3. General Bell-Shaped Membership Function

The general bell-shaped secondary MF, shown in Figure 5 is obtained via (31), where $\tilde{u}(x, u)$ is a membership function which takes on the parameters $x \in X$ on the Universe of Discourse, and $u \in U$ such that $u \in J_x \subseteq [0, 1]$. Here p_x and σ_u are the obtained parameters from the baseline Gaussian secondary membership function, and b_u controls the slope of the membership function. These parameters are obtained by (18) and (20), or (21) and (23), depending on the chosen primary membership function,

$$\tilde{\mu}(x, u) = \frac{1}{1 + \left[\left(\frac{u - p_x}{\sigma_u}\right)^2\right]^{b_u}}. \tag{31}$$

Figure 5. Sample general bell-shaped membership function.

3.3.4. Trapezoidal Membership Function

The previously shown secondary membership functions have relied on a baseline secondary Gaussian membership function, as expressed by (18) and (20), or (21) and (23). But for a trapezoidal secondary membership function, these equations are not used; instead, (32) and (33) define the required modifiers for the membership function. The trapezoidal secondary membership function parameters are obtained through (34)–(37). Here a_μ, b_μ, c_μ and d_μ are the used parameters to construct the trapezoidal secondary membership function and w_l in (35) and w_r in (36) are the left and right width,

$$\delta_l = p_x - \underline{\mu}(x), \tag{32}$$

$$\delta_r = \bar{\mu}(x) - p_x, \tag{33}$$

$$a_\mu = \underline{\mu}(x), \tag{34}$$

$$b_\mu = p_x - w_l \delta_l, \tag{35}$$

$$c_\mu = p_x + w_r \delta_r, \tag{36}$$

$$d_\mu = \bar{\mu}(x). \tag{37}$$

The trapezoidal secondary membership function, shown in Figure 6 is obtained via (38), where $\tilde{u}(x, u)$ is a membership function which takes on the parameters $x \in X$ on the Universe of Discourse, and $u \in U$ such that $u \in J_x \subseteq [0, 1]$. Here a_u, b_u, c_u and d_u are the obtained by using (34)–(37),

$$\tilde{\mu}(x, u) = \max\left(\min\left(\frac{u - a_\mu}{b_\mu - a_\mu}, 1, \frac{d_\mu - u}{d_\mu - c_\mu}\right), 0\right). \tag{38}$$
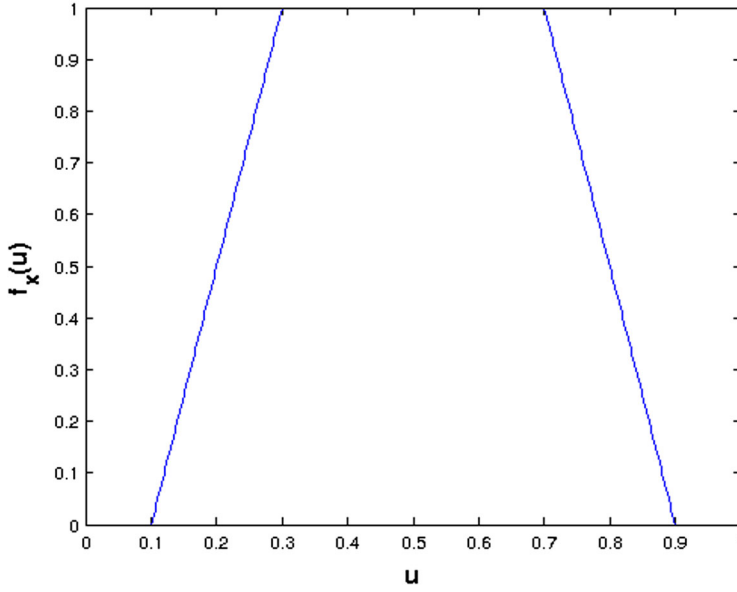
Figure 6. Sample secondary trapezoidal membership function.

3.4. GT2 FS Parameterization

Parameterization is performed in an algorithmic manner, where depending on the chosen primary membership function and secondary membership function, different equations are combined in order to construct the final GT2 membership function. Also, depending on the selected secondary membership function it may or may not require an intermediate baseline secondary membership function.

3.4.1. Gaussian Primary Membership Function with Uncertain Mean and Double Gaussian Secondary Membership Function

This GT2 membership function, defined as $\tilde{\tilde{\mu}}(x, u)$, is shown in (39) in functional form, where 'gaussmgauss2type2' stands for Gaussian primary membership function with uncertain mean and double Gaussian secondary membership function. It requires 5 parameters $\{\sigma, m_1, m_2, \lambda, \rho\}$ where σ is the standard deviation of the primary membership function, m_1 and m_2 are the left and right means of the Gaussian membership function with uncertain mean, λ and ρ are fractions of uncertainty, which affect the core and support, respectively, of the secondary membership function. In Figure 7 this constructed GT2 FS can be visualized from various angles,

$$\tilde{\tilde{\mu}}(x, u) = gaussmgauss2type2 (x, u, [\sigma, m_1, m_2, \lambda, \rho]). \tag{39}$$

The following algorithm summarizes the sequence of equations, which construct this GT2 membership function.

Procedure gaussmgauss2type2$(x, u, [\sigma, m_1, m_2, \lambda, \rho])$:

(1) Calculate the primary membership function support via (12)–(15).
(2) Calculate baseline secondary membership function via (18)–(20).
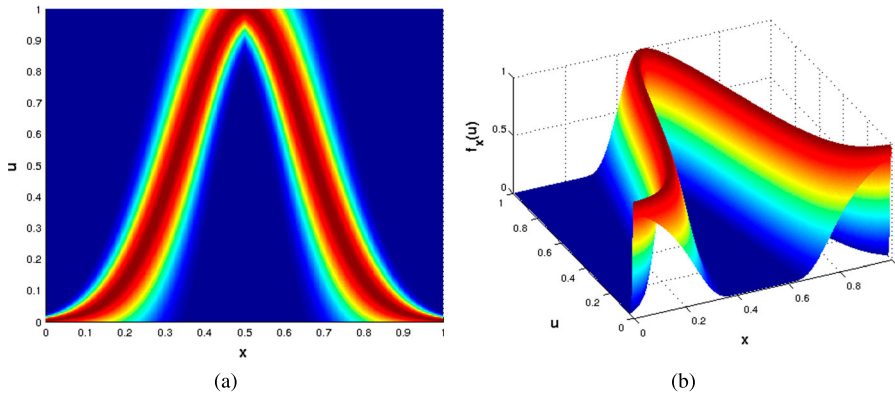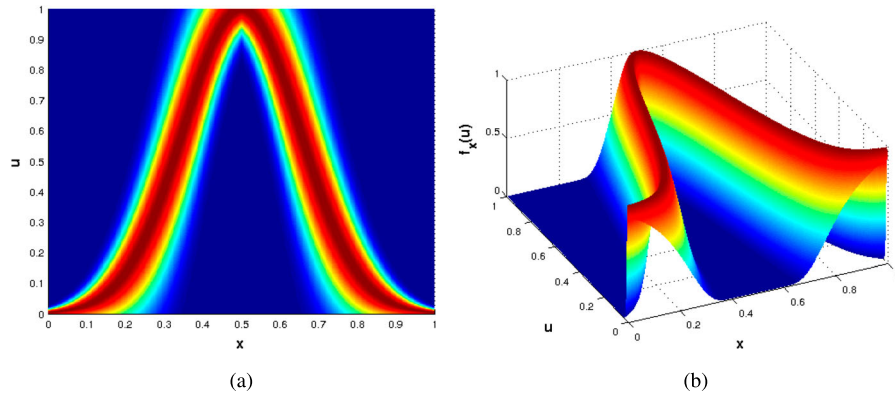
Figure 7. Sample GT2 FS constructed from a Gaussian primary membership function with uncertain mean and double Gaussian secondary membership function. (a) is a top view and (b) an isometric view.

(3) Calculate secondary membership function parameters via (26)–(29).
(4) Calculate secondary membership function via (30).
(5) Return $\tilde{\tilde{\mu}}(x, u)$.

3.4.2. Gaussian Primary Membership Function with Uncertain Mean and Gaussian Secondary Membership Function

This GT2 membership function, defined as $\tilde{\tilde{\mu}}(x, u)$, is shown in (40) in functional form, where 'gaussmgausstype2' stands for a Gaussian primary membership function with uncertain mean and Gaussian secondary membership function. It requires four parameters $\{\sigma, m_1, m_2, \rho\}$ where σ is the standard deviation of the primary membership function, m_1 and m_2 are the left and right means of the Gaussian membership function with uncertain mean, and ρ is a fraction of uncertainty which affects the support of the secondary membership function. In Figure 8 this constructed GT2 FS can be visualized from various

Figure 8. Sample GT2 FS constructed from a Gaussian primary membership function with uncertain mean and Gaussian secondary membership function. (a) is a top view and (b) an isometric view.

angles.

$$\tilde{\tilde{\mu}}(x, u) = gaussmgausstype2(x, u, [\sigma, m_1, m_2, \rho]). \qquad (40)$$

The following algorithm summarizes the sequence of equations, which construct this GT2 membership function.

Procedure gaussmgausstype2($x, u, [\sigma, m_1, m_2, \rho]$):

(1) Calculate the primary membership function support via (12)–(15).
(2) Calculate secondary membership function parameters via (18), (19) and (24).
(3) Calculate secondary membership function via (25).
(4) Return $\tilde{\tilde{\mu}}(x, u)$.

3.4.3. Gaussian Primary Membership Function with Uncertain Mean and General Bell-shaped Secondary Membership Function

This GT2 membership function, defined as $\tilde{\tilde{\mu}}(x, u)$, is shown in (41) in functional form, where 'gaussmgbelltype2' stands for Gaussian primary membership function with uncertain mean and general bell-shaped secondary membership function. It requires four parameters $\{\sigma, m_1, m_2, b_u\}$ where σ is the standard deviation of the primary membership function, m_1 and m_2 are the left and right means of the Gaussian membership function with uncertain mean and b_u controls the slope of the general bell-shaped secondary membership function. In Figure 9 this constructed GT2 FS can be visualized from various angles,

$$\tilde{\mu}(x, u) = gaussmgbelltype2\left(x, u, \left[\sigma, m_1, m_2, b_u\right]\right). \qquad (41)$$

The following algorithm summarizes the sequence of equations, which constructs this GT2 membership function.

Procedure gaussmgbelltype2($x, u, [\sigma, m_1, m_2, b_u]$):

(1) Calculate the primary membership function support via (12)–(15).
(2) Calculate secondary membership function parameters via (18)–(20).
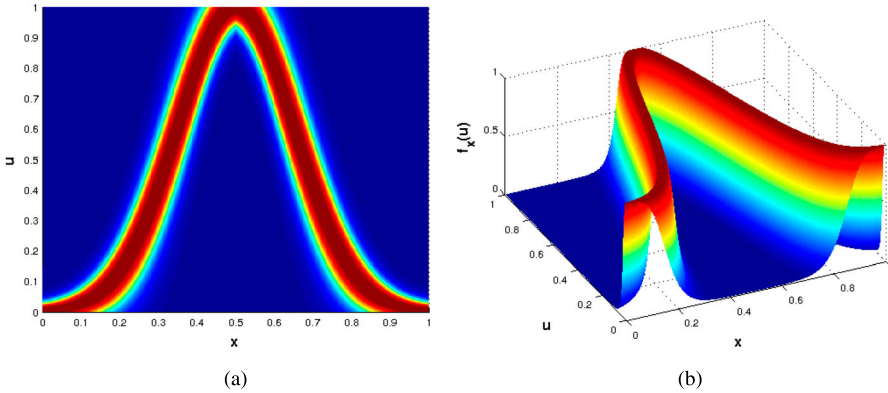

(a)   (b)

Figure 9. Sample GT2 FS constructed from a Gaussian primary membership function with uncertain mean and general bell-shaped secondary membership function. (a) is a top view and (b) an isometric view.

(3)　Calculate secondary membership function via (31).
(4)　Return $\tilde{\tilde{\mu}}(x, u)$.

3.4.4. Gaussian Primary Membership Function with Uncertain Mean and Trapezoidal Secondary Membership Function

This GT2 membership function, defined as $\tilde{\tilde{\mu}}(x, u)$, is shown in (42) in functional form, where 'gaussmtraptype2' stands for Gaussian primary membership function with uncertain mean and trapezoidal secondary membership function. It requires five parameters $\{\sigma, m_1, m_2, w_l, w_r\}$ where σ is the standard deviation of the primary membership function, m_1 and m_2 are the left and right means of the Gaussian membership function with uncertain mean, w_l and w_r provide the width of the core of the trapezoidal secondary membership function. In Figure 10 this constructed GT2 FS can be visualized from various angles,

$$\tilde{\tilde{\mu}}(x, u) = gaussmtraptype2\left(x, u, [\sigma, m_1, m_2, w_l, w_r]\right). \tag{42}$$

The following algorithm summarizes the sequence of equations, which construct this GT2 membership function.

Procedure gaussmtraptype2($x, u, [\sigma, m_1, m_2, w_l, w_r]$):

(1)　Calculate the primary membership function support via (12)–(15).
(2)　Calculate secondary membership function parameters via (32)–(37).
(3)　Calculate secondary membership function via (38).
(4)　Return $\tilde{\tilde{\mu}}(x, u)$.

3.4.5. Gaussian Primary Membership Function with Uncertain Standard Deviation and Double Gaussian Secondary Membership Function

This GT2 membership function, defined as $\tilde{\tilde{\mu}}(x, u)$, is shown in (43) in functional form, where 'gausssgauss2type2' stands for Gaussian primary membership function with uncertain standard deviation and double Gaussian secondary membership function. It requires five
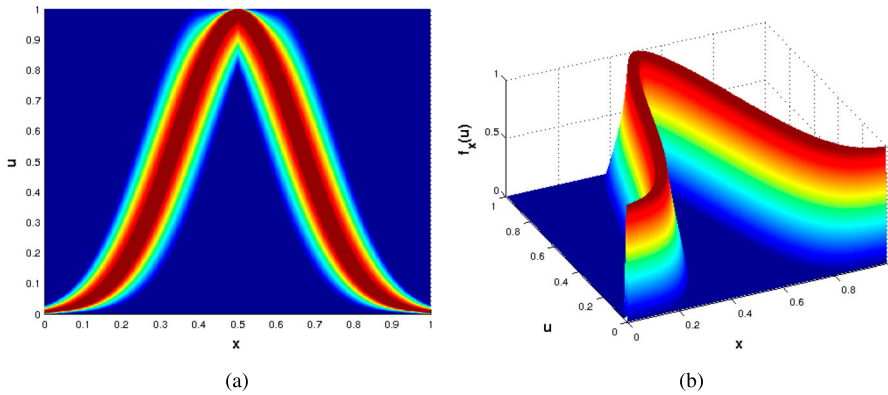

(a)　　　　　　　　　　　　　　(b)

Figure 10. Sample GT2 FS constructed from a Gaussian primary membership function with uncertain mean and trapezoidal secondary membership function. (a) is a top view and (b) an isometric view.
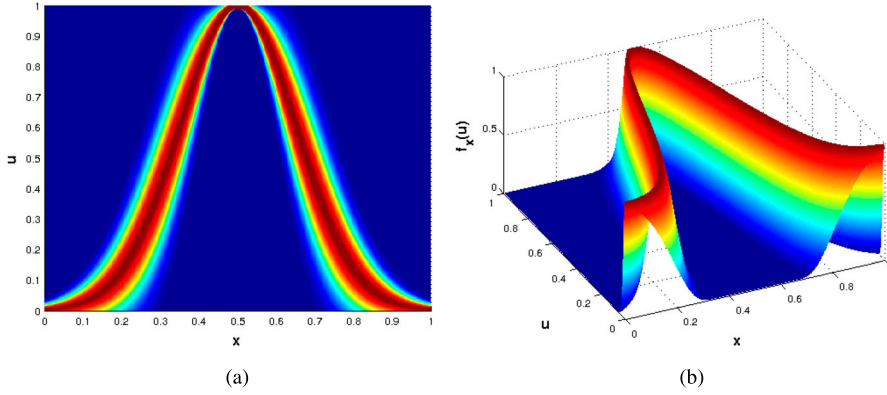
Figure 11. Sample GT2 FS constructed from a Gaussian primary membership function with uncertain standard deviation and double Gaussian secondary membership function. (a) is a top view and (b) an isometric view.

parameters $\{\sigma_1, \sigma_2, m, \lambda, \rho\}$ where σ_1 and σ_2 are the inner and outer standard deviations of the primary membership function with uncertain standard deviation, m is the mean of the primary Gaussian membership function, λ and ρ are fractions of uncertainty which affect the core and support, respectively, of the secondary membership function. In Figure 11 this constructed GT2 FS can be visualized from various angles,

$$\tilde{\tilde{\mu}}(x, u) = gausssgauss2type2 \left(x, u, [\sigma_1, \sigma_2, m, \lambda, \rho] \right). \tag{43}$$

The following algorithm summarizes the sequence of equations, which constructs this GT2 membership function.

Procedure gausssgauss2type2($x, u, [\sigma_1, \sigma_2, m, \lambda, \rho]$):

(1) Calculate the primary membership function support via (16) and (17).
(2) Calculate baseline secondary membership function via (21)–(23).
(3) Calculate secondary membership function parameters via (26)–(29).
(4) Calculate secondary membership function via (30).
(5) Return $\tilde{\tilde{\mu}}(x, u)$.

3.4.6. Gaussian Primary Membership Function with Uncertain Standard Deviation and Gaussian Secondary Membership Function

This GT2 membership function, defined as $\tilde{\tilde{\mu}}(x, u)$, is shown in (44) in functional form, where 'gausssgausstype2' stands for Gaussian primary membership function with uncertain standard deviation and Gaussian secondary membership function. It requires four parameters $\{\sigma_1, \sigma_2, m, \rho\}$ where σ_1 and σ_2 are the inner and outer standard deviations of the primary membership function with uncertain standard deviation, m is the mean of the primary membership function, ρ is a fraction of uncertainty which affects the support. In Figure 12 this constructed GT2 FS can be visualized from various angles,

$$\tilde{\tilde{\mu}}(x, u) = gausssgausstype2 \left(x, u, [\sigma_1, \sigma_2, m, \rho] \right). \tag{44}$$

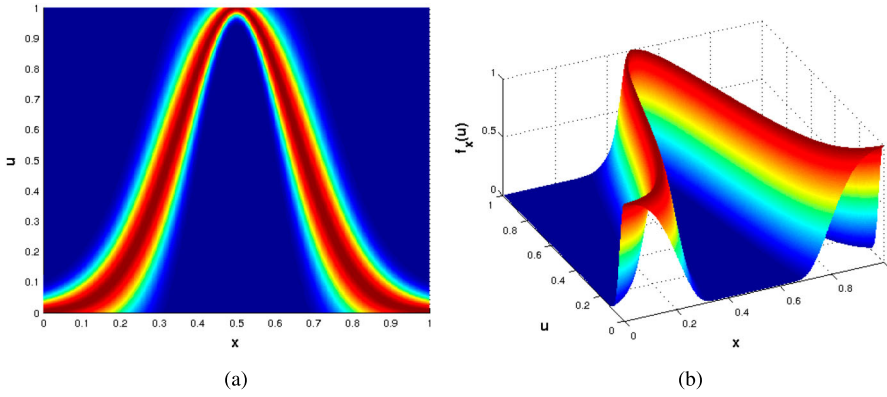(a)   (b)

Figure 12. Sample GT2 FS constructed from a Gaussian primary membership function with uncertain standard deviation and a Gaussian secondary membership function. (a) is a top view and (b) an isometric view.

The following algorithm summarizes the sequence of equations, which constructs this GT2 membership function.

Procedure gausssgausstype2($x, u, [\sigma, m_1, m_2, \rho]$):

(1) Calculate the primary membership function support via (16) and (17).
(2) Calculate secondary membership function parameters via (21), (22) and (24).
(3) Calculate secondary membership function via (25).
(4) Return $\tilde{\tilde{\mu}}(x, u)$.

3.4.7. Gaussian Primary Membership Function with Uncertain Standard Deviation and General Bell-Shaped Secondary Membership Function

This GT2 membership function, defined as $\tilde{\tilde{\mu}}(x, u)$, is shown in (45) in functional form, where 'gausssgbelltype2' stands for Gaussian primary membership function with uncertain standard deviation and general bell-shaped secondary membership function. It requires four parameters $\{\sigma_1, \sigma_2, m, b_u\}$ where σ_1 and σ_2 are the inner and outer standard deviations of the Gaussian primary membership function with uncertain standard deviation, m is the mean of the primary membership function, and b_u controls the slope of the general bell-shaped secondary membership function. In Figure 13(a,b) this constructed GT2 FS can be visualized from various angles,

$$\tilde{\tilde{\mu}}(x, u) = gausssgbelltype2\left(x, u, [\sigma_1, \sigma_2, m, b_u]\right). \tag{45}$$

The following algorithm summarizes the sequence of equations, which constructs this GT2 membership function.

Procedure gausssgbelltype2($x, u, [\sigma_1, \sigma_2, m, b_u]$):

(1) Calculate the primary membership function support via (16) and (17).
(2) Calculate secondary membership function parameters via (21)–(23).
(3) Calculate secondary membership function via (31).
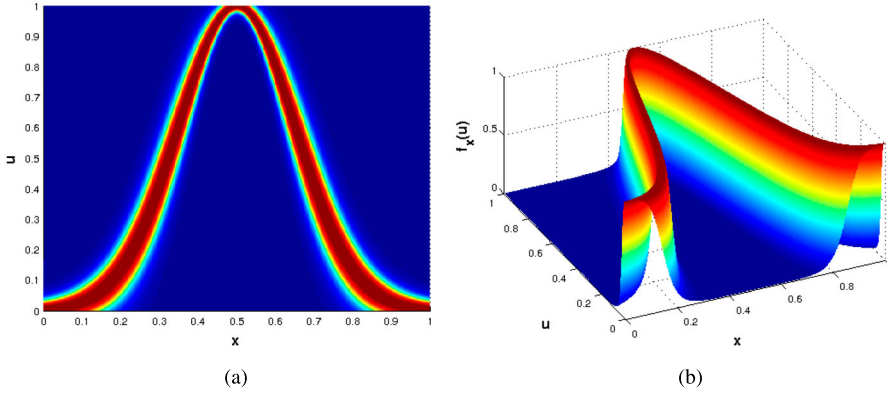(4) Return $\tilde{\tilde{\mu}}(x, u)$.

Figure 13. Sample GT2 FS constructed from a Gaussian primary membership function with uncertain standard deviation and general bell-shaped secondary membership function. (a) is a top view and (b) an isometric view.

3.4.8. Gaussian Primary Membership Function with Uncertain Standard Deviation and Trapezoidal Secondary Membership Function

This GT2 membership function, defined as $\tilde{\tilde{\mu}}(x, u)$, is shown in (46) in functional form, where 'gaussstraptype2' stands for Gaussian primary membership function with uncertain standard deviation and trapezoidal secondary membership function. It requires five parameters $\{\sigma_1, \sigma_2, m, w_l, w_r\}$ where σ_1 and σ_2 are the inner and outer standard deviations of the Gaussian primary membership function with uncertain standard deviation, m is the mean of the primary membership function, w_l and w_r define the width of the core of the trapezoidal secondary membership function. In Figure 14 this constructed GT2 FS can be visualized from various angles,

$$\tilde{\tilde{\mu}}(x, u) = gaussstraptype2\left(x, u, [\sigma_1, \sigma_2, m, w_l, w_r]\right). \tag{46}$$

The following algorithm summarizes the sequence of equations, which constructs this GT2 membership function.
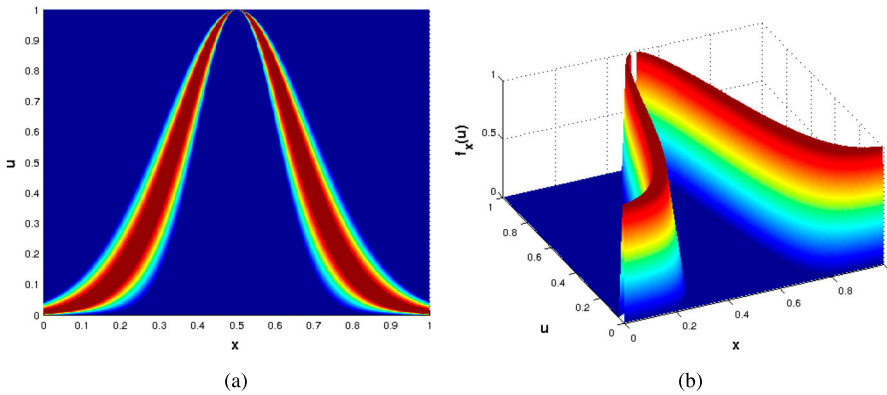

Figure 14. Sample GT2 FS constructed from a Gaussian primary membership function with uncertain standard deviation and trapezoidal secondary membership function. (a) is a top view and (b) an isometric view.

Procedure gaussstraptype2$(x, u, [\sigma_1, \sigma_2, m, w_l, w_r])$:

(1) Calculate the primary membership function support via (16) and (17).
(2) Calculate secondary membership function parameters via (21)–(23).
(3) Calculate secondary membership function via (31).
(4) Return $\tilde{\tilde{\mu}}(x, u)$.

4. Methodology and Results

In this section, we present the methodology and simulation results for three different applications: Mackey–Glass chaotic series prediction, water tank controller and the wave equation approximation. The main goal is to analyze the performance of these systems when the parameterization model of GT2 membership functions proposed in Section 3 is applied; since this is the main contribution of this paper. The applications were designed using different types of membership functions and the GT2 fuzzy inference system was approximated using the α-plane theory.

In order to evaluate the accuracy of the proposed fuzzy systems, the Root Mean Square Error (RMSE) metric is used; additionally, the results are compared with IT2 FSs. The IT2 FSs are designed under the same conditions than the GT2 FSs, as the number of antecedents, consequents and fuzzy rules, type of MFs, type inference and type-reduction.

It is important to mention that the idea to compare the results achieved by the GT2 FSs against the IT2 FSs is only to show that the proposed parameterization method is a good technique, which can be applied to improve results on any application. The obtained results are described in detail below.

4.1. Mackey–Glass Time Series

The Mackey–Glass Time Series is a chaotic time series that was proposed by Mackey and Glass [44]. The prediction of future values of these time series is a benchmark problem, which is used in this paper as an application. The time series is obtained from the following non-linear equation expressed in (47). In the case where $\tau > 17$, it is known to exhibit chaotic behavior,

$$x'(t) = \frac{0.2x\,(t - \tau)}{1 + x^{10}\,(t - \tau)} - 0.1x(t). \tag{47}$$

The fourth-order Runge–Kutta method is used to obtain the numerical solutions to the equation. From the Mackey–Glass time series $x(t)$, we extracted 1000 input-output data points from $t = 124$ to $t = 1123$, with an initial condition $x(0) = 1.2$, $\tau = 17$, and time step $= 0.1$ where $x(t)$ is derived for $0 \leq t \leq 1200$ with an initial condition pairs of the following format

$$[x\,(t - 24), x\,(t - 18), x\,(t - 12), x\,(t - 6); x(t)].$$

The 1000 data points used in the prediction problem are presented in Figure 15. The first 500 data points are used for training the fuzzy predictor while the remaining 500 data points are used for validating the prediction performance of the proposed model.
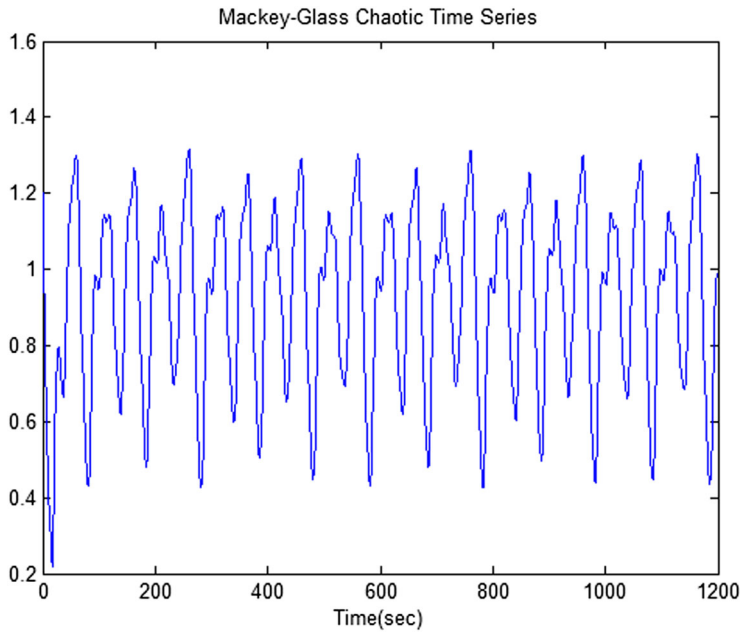
Figure 15. Mackey–Glass Chaotic Time Series.

4.1.1. Mackey–Glass Time Series Using GT2 FSs

The GT2 fuzzy system predictor for the Mackey–Glass chaotic time series is a Singleton Takagi–Sugeno fuzzy model. The fuzzy model has four inputs and one output; each input is granulated into two membership functions, which are labeled with *low* and *high* linguistic values, as shown in Figure 16. The input membership functions are Gaussian primary membership functions with uncertain standard deviation and general bell-shaped secondary membership function (gausssgbelltype2) and the parameterization for this type of membership function is expressed in (45).

The GT2 fuzzy predictor is designed using 16 fuzzy rules, which are characterized by the number of antecedents fuzzy sets combined with the consequents fuzzy sets.

4.1.2. Mackey–Glass Time-Series Results

In the first experiment we considered the Mackey–Glass time-series prediction with $\tau = 17$ and in a noiseless situation, using GT2 FS and IT2 FS. In Table 1, we can note that the RMSE
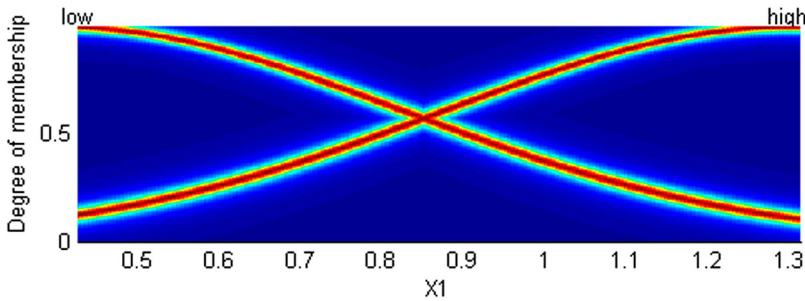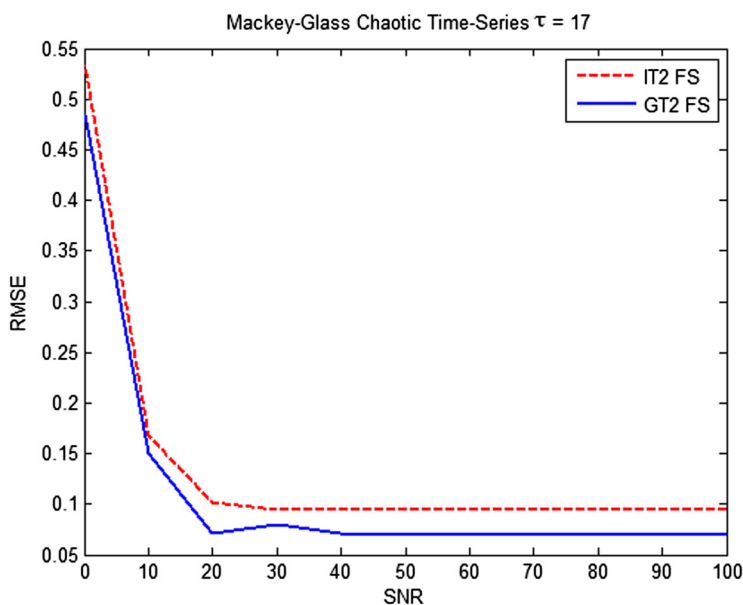

Figure 16. GT2 membership functions for the input X1.

Table 1. The noise-free Mackey–Glass chaotic time-series prediction.

Fuzzy system	RMSE $\tau = 17$
GT2 FSs	0.0704
IT2 FSs	0.0943

Table 2. Mackey–Glass chaotic time-series prediction corrupted by uniformly-distributed stationary additive noise.

Fuzzy system	RMSE				
	SNR (0 dB)	SNR (10 dB)	SNR (20 dB)	SNR (30 dB)	SNR (40 dB)
GT2FSs	0.4848	0.1503	0.0803	0.0708	0.0706
IT2FSs	0.5332	0.1678	0.1024	0.095	0.0945

Figure 17. Mackey–Glass time-series prediction for $\tau = 17$ corrupted with noise using IT2 and GT2 FS.

achieved by the GT2 FS is better than the IT2 FSs with a value of 0.0704, which improved the fuzzy prediction system.

In the second experiment, the Mackey–Glass time-series prediction for $\tau = 17$ is now corrupted with noise levels, we add values of 0 dB, 10 dB, 20 dB and 30 dB of SNR (signal noise ratio) as a high source of uncertainty. In this case, Table 2 shows the RMSE achieved by using GT2 FS and IT2 FS when the different noise levels are applied. In Figure 17 we can note that GT2 FS performs better than IT2 FS. This is due to the fact that GT2 FS can handle uncertainty better and thus increase the learning ability.

4.2. Water Tank Controller

The water tank is a benchmark problem, which is used for controlling the water level in a tank [45]. In designing the control system, the valve opening size and speed are determined

according to the level and rate of water input flow. The controller has to be able to set the valve at the correct position to maintain the liquid level accurately for a given desired value.

To evaluate the valve opening in a precise way, a GT2 Fuzzy controller is implemented in this test.

4.2.1. Water Tank Controller Using GT2 FSs

The water tank GT2 fuzzy system has two input variables, the first one is called *Level*, which has three membership functions with the *low*, *okay* and *high* linguistic values, as shown in Figure 18. The second input variable is called *Rate* with three membership functions, which are represented by the linguistic values of *negative*, *good* and *positive*, illustrated in Figure 18.

The GT2 fuzzy system has one output labeled as *Valve*, which has five membership functions with the *closefast*, *closeslow*, *nochange*, *openslow* and *openfast* linguistic values, as shown in Figure 19.

The input and output membership functions are defined by Gaussian primary membership functions with uncertain mean and Gaussian secondary membership functions (*gaussmgausstype2*); the parameterization for these membership functions are expressed in (40).

For modeling the knowledge about the problem with the MamdaniGT2 fuzzy system, we consider five fuzzy rules, which are detailed below

(1) If (level is okay) then (valve is nochange)
(2) If (level is low) then (valve is openfast)
(3) If (level is high) then (valve is closefast)
(4) If (level is okay) and (rate is positive) then (valve is closeslow)
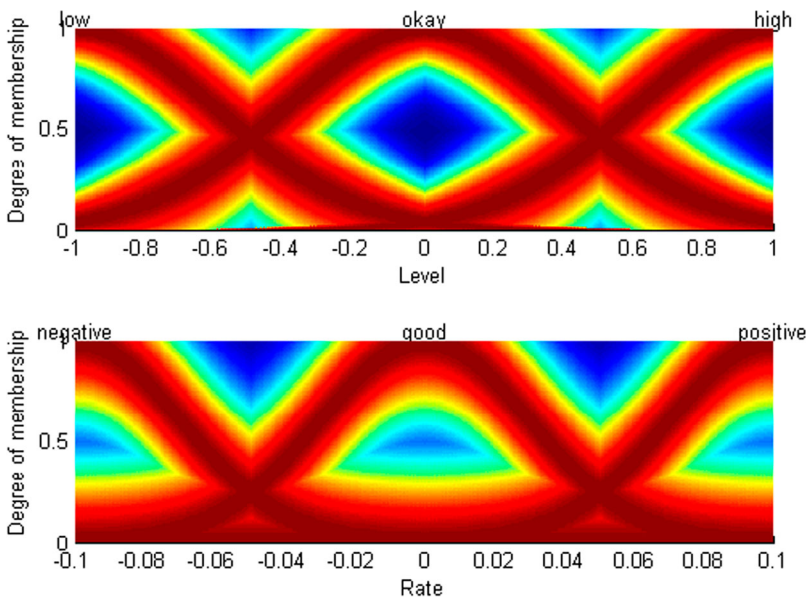(5) If (level is okay) and (rate is negative) then (valve is openslow)

Figure 18. The input GT2 membership functions for the variables 'Level' and 'Rate'.
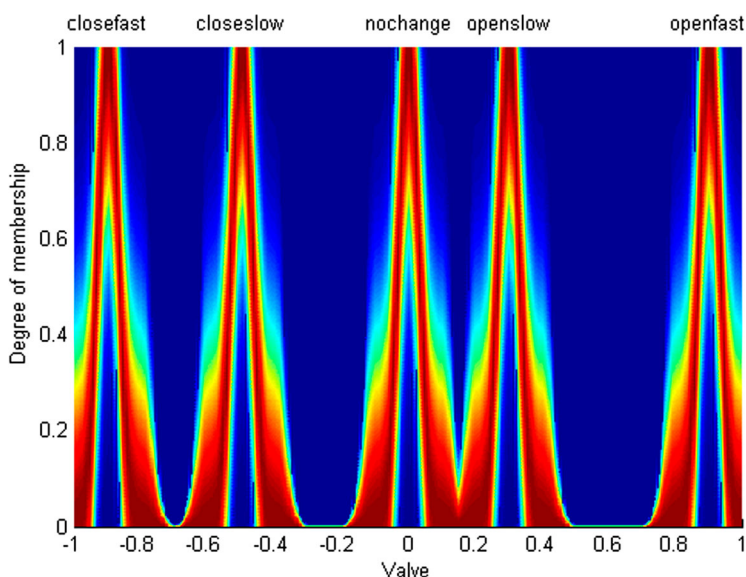
Figure 19. The output GT2 membership function for variable 'Valve'.

Table 3. RMSE results for the water tank fuzzy system controller.

	RMSE		
Fuzzy system	Level = 0.5	Level = 0.4	Level = 0.3
GT2 FSs	0.4058	0.3262	0.2612
IT2 FSs	0.4172	0.5076	0.6010

4.2.2. Water Tank Controller Results

In the first simulation for the water tank GT2 fuzzy system, we considered different reference values for the water tank level (0.5, 0.4 and 0.3). According to the results presented in Table 3, the RMSE achieved by the GT2 fuzzy controller performed better than the IT2 FS.

4.3. The Wave Equation

An eigenfunction of the wave equation is a mathematical model of how a disturbance travels through matter [46]. If t is time and x and y are spatial coordinates with the units chosen so that the wave propagation speed is equal to one, then the amplitude of a wave satisfies the partial differential equation expressed as follows:

$$\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2. \tag{48}$$

In the case of periodic time behavior, this gives solutions of the form

$$u(t, x, y) = \sin(\sqrt{\lambda}t)v(x, y), \tag{49}$$

where

$$\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 + \lambda v = 0. \tag{50}$$

In (49) the parameter λ represents the eigenvalues and the functions $v(x, y)$ are the eigenfunctions or modes of vibration. They are determined by the physical properties, the geometry and the boundary conditions of each particular situation. Any solution to the wave equation can be expressed as a linear combination of these eigenfunctions. The square roots of the eigenvalues are resonant frequencies. A periodic external driving
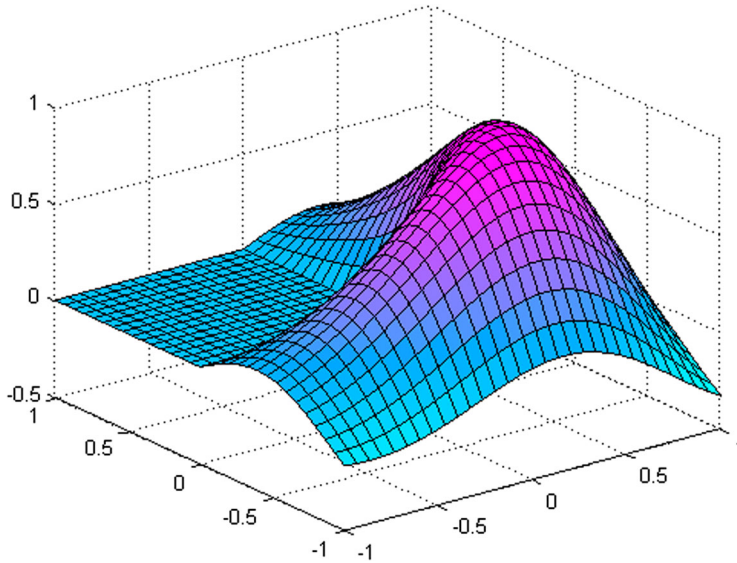

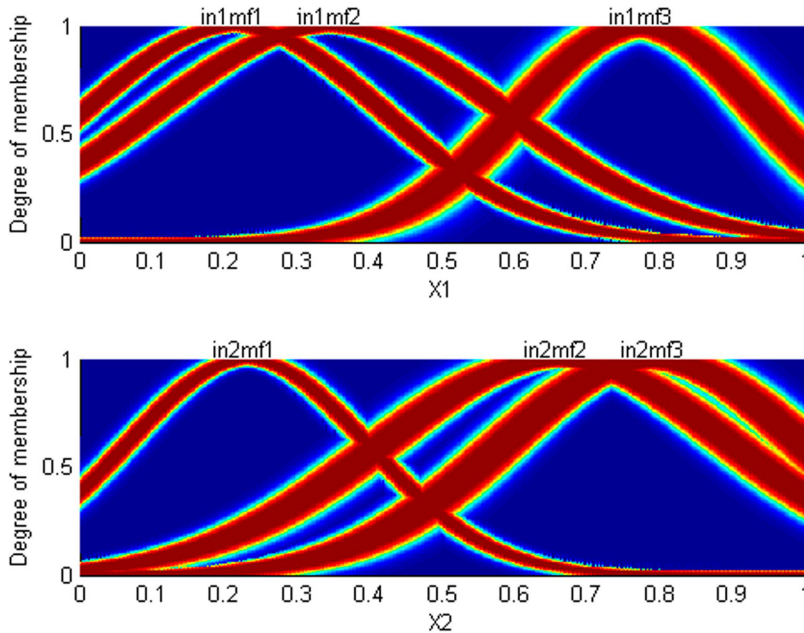**Figure 20.** Surface obtained by the membrane function.

Figure 21. The inputs GT2 membership function for variables 'X1' and 'X2'.

force at one of these frequencies will generate an unboundedly strong response in the medium.

4.3.1. Approximate Wave Equation with GT2 FS

In this case study we approximate the wave equation surface using a GT2 FS, the idea is to compare our proposed fuzzy approach against those produced by the Membrane function [45]. The surface obtained by the Membrane function is shown in Figure 20.

For the GT2 fuzzy approach, we used a Singleton Takagi–Sugeno fuzzy system model, which was designed with two inputs and one output. The inputs ($X1$ and $X2$) are granulated into three membership functions, and are defined by Gaussian primary membership functions with uncertain mean and general bell-shaped secondary membership function (*gaussmgbelltype2*), as shown in Figure 21; the parameterization for these membership functions are expressed in (41). The knowledge base of the GT2 FS is implemented using nine fuzzy rules, which are characterized by the number of antecedents fuzzy sets combined with the consequents fuzzy sets. The RMSE (0.0913) and correlation (0.9690) obtained by the GT2 FSs with respect to the MATLAB® Membrane function are presented in Table 4. The output surface obtained by the GT2 FS is shown in Figure 22 and we can note that this surface is very similar to that obtained in Figure 20; therefore, the parameterization approach proposed in this paper is a good way to design a GT2 FS.

Table 4. RMSE and correlation coefficient results for the wave equation fuzzy approximation.

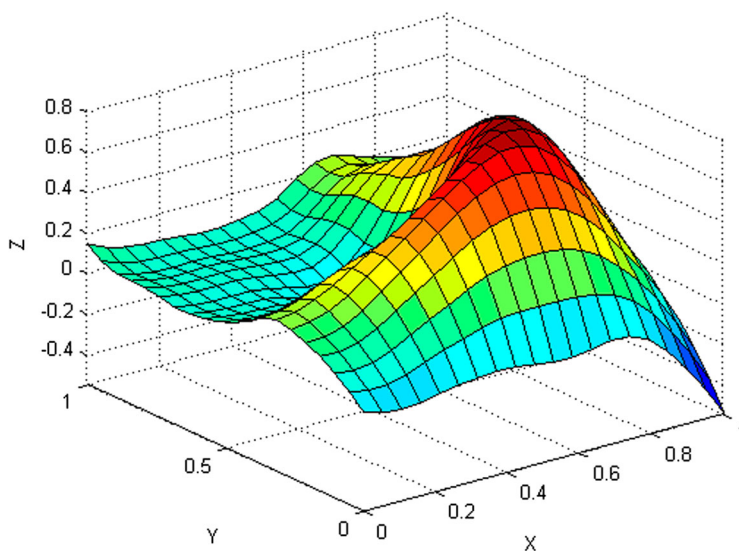| Fuzzy system	RMSE	r
GT2 FSs	0.0913	0.9690

Figure 22. Surface of the wave equation using GT2 FS.

5. Conclusion

In this paper, we have presented a new approach for the parameterization of general type-2 fuzzy sets. With this approach, very few parameters are needed to build different membership function types. Examples were given of possible combinations between primary and secondary membership functions, and although only a small amount of examples were given, the concepts behind the proposed parameterization can easily be used to construct any other combination of GT2 membership functions.

Illustrations were also presented for all given combinations of primary and secondary membership functions, which demonstrate that the proposed parameterization can form smooth GT2 membership functions and these are recognizable since they are an extension from IT2 membership functions.

The presented experimentation demonstrated that the use of the proposed parameterization in several benchmark datasets is a good choice in designing general type-2 fuzzy systems. Simulation results show that GT2 FSs achieved better prediction accuracy than IT2 FSs in the considered benchmark problems.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

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